Convective Instability in Porous Media during Solidification

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In this study, the convective instability in porous media during solidification is analyzed using the linear stability theory. The governing equation in the porous media is based on the Darcy's equation. The present predictions for the onset of convection recover the results of convective instability in packed beds with through flow for large Peclet number.

Solidification of porous media has its applications in engineering practices with phase change, such as materials processing, thermal energy storage, and so on. When a pure melt in porous media is solidified from above, thermal convection can be induced by an unstable density profile in the porous media under a gravitational field. Convective instability problems during solidification of a liquid have been studied extensively, while those of porous media have received less attention. Recent studies (Karcher and Müller, 1995; Mackie et al., 1999) investigated the Rayleigh-Benard stability problem with solidification of porous media. Smith (1988) examined the onset of convection driven by the thermal gradient under the solidifying interface during directional solidification of a pure liquid, whereas the present study is concerned with that of porous media.

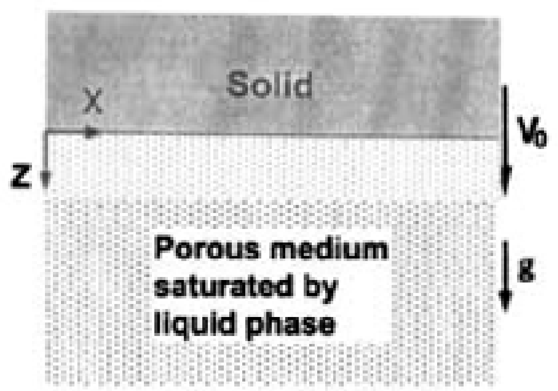

Figure 1. Solidification of porous media.

Governing equations

The porous media saturated with a pure liquid is solidified downward in a semi-infinite region, as shown in Figure 1. The coordinate system is attached to the planar interface that is moving at the constant velocity V_0. Under the assumption that the thermophysical properties of the liquid phase and the porous material, such as heat capacity and thermal conductivity, are the same, the governing equations are given as follows

For the solid layer

$$\frac{\partial T_S}{\partial t} - V_0 \frac{\partial T_S}{\partial Z} = \alpha_S \nabla^2 T_S, \tag{1}$$

For the porous media

$$\frac{\partial T_L}{\partial t} - V_0 \frac{\partial T_L}{\partial Z} + \boldsymbol{U}\cdot\nabla T_L = \alpha_L \nabla^2 T_L, \tag{2}$$

$$\frac{\mu}{K}\boldsymbol{U} = -\nabla P + \rho_L(1 - \beta(T_L - T_0))g\boldsymbol{e}_k, \tag{3}$$

$$\nabla\cdot\boldsymbol{U} = 0, \tag{4}$$

where T denotes the temperature, t the time, Z the vertical coordinate, $\boldsymbol{U}$ the velocity, and P the pressure. α denotes the thermal diffusivity, μ the viscosity, K the permeability, ρ the density, β the thermal expansion coefficient, and g the gravity acceleration. The subscript S and L represent the solid and liquid quantities, respectively. T_0 denotes the reference temperature and $\boldsymbol{e}_k$ the unit vector in the Z-direction. Equation 3 is the Darcy's equation, which is under the Boussinesq approximation. The density change in the liquid can induce buoyancy-driven convection. In the present study, solidification shrinkage is neglected, and the densities of the solid and the liquid are assumed to be the same. The simple boundary conditions are given as follows

$$W = 0 \tag{5a}$$

$$T_S = T_L = T_M \tag{5b}$$

$$k_S \frac{dT_S}{dZ} - k_L \frac{dT_L}{dZ} = LV_0 \tag{5c}$$

at $Z = 0$,

$$W = 0 \tag{6a}$$

$$T_L = T_\infty \tag{6b}$$

for $Z \to \infty$

where W denotes the vertical velocity, k the thermal conductivity, and L the latent heat of fusion. T_M denotes the melting temperature at the solidification interface and T_∞ denotes the temperature in the porous media far from the interface. Using Eqs. 2, 5b, and 6b, the basic thermal profile in the porous media with no motion is given by the form of $\overline{T_L} = T_M + (T_\infty - T_M)(1 - \exp(-V_0 Z/\alpha_L))$.

The linear stability theory assumes that infinitesimally small perturbed quantities are enforced into the basic state. Under the normal modes analysis, the following linearized dimensionless disturbance equations are obtained from the governing Eqs. 1–4

For the solid layer

$$(D^2 + \alpha_r^{-1} D - a^2)\theta_S = 0, \tag{7}$$

For the porous media

$$(D^2 + D - a^2)\theta_L = w\exp(-z), \tag{8}$$

$$(D^2 - a^2)w = a^2 R\theta_L, \tag{9}$$

where $D = d/dz$, $z = V_0 Z/\alpha_L$, and $\alpha_r = \alpha_S/\alpha_L$. The dimensionless disturbance temperature θ is scaled by ΔT ($= T_\infty - T_M$) and the dimensionless disturbance vertical velocity w by V_0. The Darcy-Rayleigh number, based on an effective depth α_L/V_0, is defined as $R = \rho_L g \beta \Delta T K / (\mu V_0)$, and a is the dimensionless horizontal wave number. The following boundary conditions are applied to the disturbance Eqs. 7–9

$$\theta_S = 0 \tag{10}$$

for $z \to -\infty$,

$$w = 0 \tag{11a}$$

$$\theta_S = \theta_L \tag{11b}$$

$$k_r D\theta_S = D\theta_L \tag{11c}$$

at $z = 0$,

$$w = 0 \tag{12a}$$

$$\theta_L = 0 \tag{12b}$$

for $z \to \infty$

where $k_r = k_s/k_L$ is the thermal conductivities ratio of solid and liquid.

Results and Discussion

The disturbance Eq. 7 for the solid layer can be solved independent of the disturbance equations for the porous media with the boundary conditions Eqs. 10 and 11b, resulting in $\theta_S = \theta_{L|_{z=0}} \exp(((-1+\sqrt{1+4\alpha_r^2 a^2})/2\alpha_r)z)$ (Smith, 1988). Then, the boundary condition (Eq. 11c) becomes $D\theta_L - k_r((\sqrt{-1+4\alpha_r^2 a^2})/2\alpha_r)\theta_L = 0$ at $z = 0$. The eigenvalue problem of Eqs. 8 and 9 is solved by employing a coordinate transformation and the Frobenius method (Hurle et al., 1983). For a given k_r and α_r, the minimum value of the Darcy-Rayleigh number R_c and its corresponding value of wave number a_c are found on the marginal stability curve.

R_c and a_c-values determine the critical conditions for the onset of convection in the porous media. We consider the two extreme cases of $k_r = \infty$ and 0. Numerical results give the critical values of $R_c = 14.35$, $a_c = 0.76$ for $k_r = \infty$ and $R_c = 10.49$, $a_c = 0.71$ for $k_r = 0$. The present results can be compared with those of porous media with through flow for large Peclet number. The present critical values for $k_r = \infty$ are almost the same as the results of Homsy and Sherwood (1976) and Jones and Persichetti (1986). Jones and Persichetti's Eq. 9, that is, $[D(D^2 - a^2) + (D^2 - a^2)^2 - \lambda' a^2 \exp(-z)]\Gamma = 0$ is equivalent to Eqs. 8 and 9 of the present study. For $k_r = \infty$, the interface is a conducting boundary, that is, a constant temperature. For $k_r = 0$, an insulating boundary, that is, a

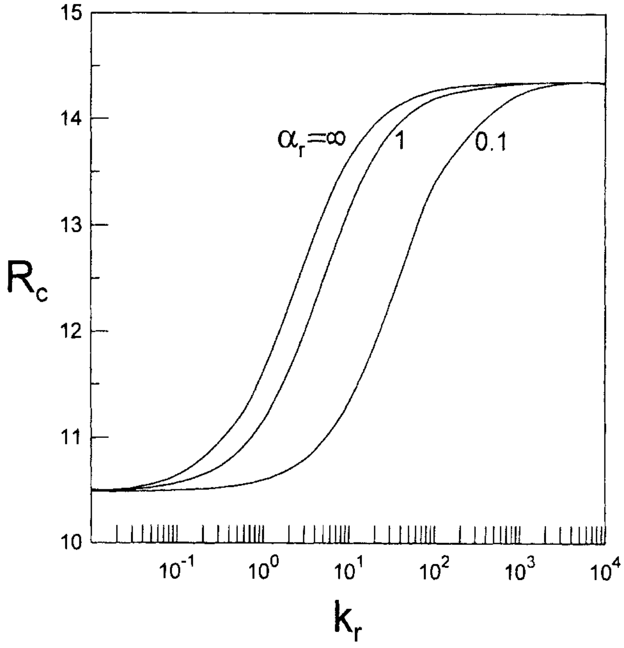

Figure 2. Effect of k_r on R_c for various values of α_r.

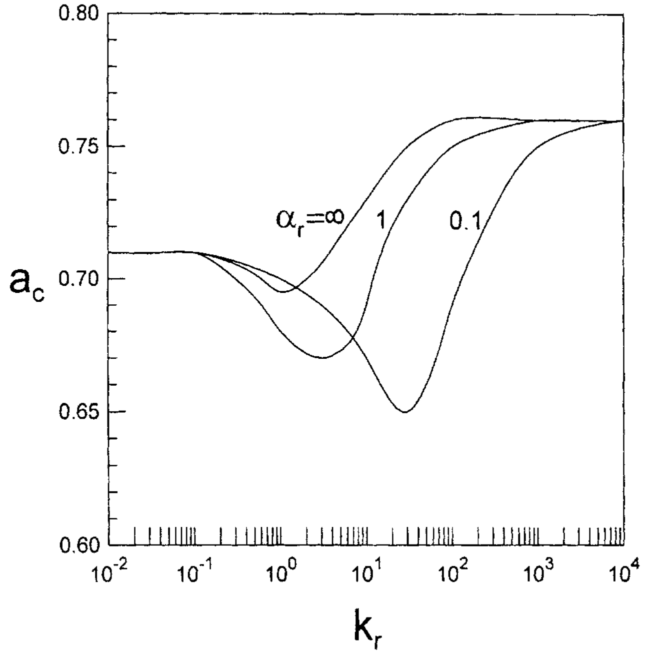

Figure 3. Effect of k_r on a_c for various values of α_r.

constant heat-flux, is applied on the interface. The classical Rayleigh-Benard problem of porous media, the so-called "Horton-Rogers-Lapwood problem," gives $R_c = 39.48$, $a_c = 3.14$ for a conducting boundary, and $R_c = 27.10$, $a_c = 2.33$ for an insulating boundary (Nield and Bejan, 1992). The present R_c-values are smaller by comparison, about 37% of the Horton-Rogers-Lapwood values.

Figures 2 and 3 show the effect of k_r on the critical conditions for various values of α_r. R_c and a_c approach an asymptotic value for $k_r \to \infty$ and 0, respectively. For $\alpha_r \to \infty$, the thermal boundary condition on the interface becomes $D\theta_L - k_r a \theta_L = 0$, and, for $\alpha_r \to 0$, that becomes $D\theta_L = 0$. So, the effect of α_r on the critical conditions is not significant for small k_r (< 0.1). With increasing k_r and α_r, the critical Darcy-Rayleigh number R_c increases and the system becomes more stable. It is found that the critical wave number a_c has a minimum point with varying k_r. The minimum point decreases with decreasing α_r. The present a_c-values of 0.67 ~ 0.76 with $\alpha_r = 1$ are larger than Smith's (1988) a_c-values of 0.35 ~ 0.6. Therefore, roll-cells appearing under the solidifying interface in porous media are smaller than those in a pure liquid. The extension of the present study, which includes the Brinkman's equation for porous media, will be investigated in the future.

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